# Riemann Surfaces, Examples 1 

Michaelmas 2020

Please send any comments or corrections to me at hjrw2@cam.ac.uk.

1. Let $U=\mathbb{C} \backslash([-1,0] \cup[1, \infty))$ and let $\gamma$ be a closed curve in $U$. Using standard properties of winding numbers, show that (i) $n(\gamma, 1)=0$, and (ii) $n(\gamma, 0)=$ $n(\gamma,-1)$.
2. Let $P\left(w_{0}, w_{1}, \ldots, w_{s} ; z\right)$ be a polynomial in the $s+1$ complex variables $w_{0}, w_{1}, \ldots, w_{s}$, where the coefficients of $P$ are holomorphic on $\mathbb{C}$. Thus

$$
P\left(f(z), f^{(1)}(z), \ldots, f^{(s)}(z) ; z\right)=0
$$

is a differential equation, which we abbreviate to $P(f)=0$. If $(f, D)$ is a function element with $P(f)=0$ in $D$ and if $\left(g, D^{\prime}\right) \approx(f, D)$ is an analytic continuation, then show that $P(g)=0$ in $D^{\prime}$. Give an example of a differential equation and function elements as above, where $D^{\prime}=D$ but $g \neq f$ on $D$.
3. Let $\pi: \tilde{X} \rightarrow X$ be a covering map of topological spaces (recalling here that the spaces are assumed connected and Hausdorff), and $f: \tilde{X} \rightarrow \tilde{X}$ a continuous map such that $\pi \circ f=\pi$. Show that $f$ has no fixed points unless it is the identity.
4. Show that the power series $f(z)=\sum_{n>1} \frac{1}{n(n-1)} z^{n}$ defines an analytic function $(1-z) \log (1-z)+z$ on the unit disc $D$. Deduce that the function element $(f, D)$ defines a complete analytic function on $\mathbb{C} \backslash\{1\}$, but does not extend to an analytic function on $\mathbb{C} \backslash\{1\}$.
5. Show that the power series $f(z)=\sum z^{2^{n}} / 2^{n}$ has the unit circle as a natural boundary.
6. Let $T$ be the complex torus $\mathbb{C} /\langle 1, \tau\rangle$, and let $Q_{1} \subset \mathbb{C}$ be the open parallelogram with vertices $0,1, \tau, 1+\tau$, and $Q_{2}$ the translation of $Q_{1}$ by $(1+\tau) / 2$. Let $U_{1}, U_{2}$ denote the open subsets of $T$ given by projection of $Q_{1}, Q_{2}$ respectively, and let $\phi_{1}: U_{1} \rightarrow Q_{1}, \phi_{2}: U_{2} \rightarrow Q_{2}$ be the charts obtained by taking the inverse maps. Describe explicitly the transition function

$$
\phi_{2} \circ \phi_{1}^{-1}: \phi_{1}\left(U_{1} \cap U_{2}\right) \rightarrow \phi_{2}\left(U_{1} \cap U_{2}\right) .
$$

7. By considering the singularity at $\infty$ or otherwise, show that any injective analytic map $f: \mathbb{C} \rightarrow \mathbb{C}$ has the form $f(z)=a z+b$, for some $a \in \mathbb{C}^{*}$ and $b \in \mathbb{C}$. Find the injective analytic maps $\mathbb{C}_{\infty} \rightarrow \mathbb{C}_{\infty}$.
8. Let $\Lambda=\left\langle\tau_{1}, \tau_{2}\right\rangle$ be a lattice in $\mathbb{C}$ and let $T=\mathbb{C} / \Lambda$ be the corresponding complex torus. Let $\Lambda^{\prime}$ denote the lattice $\left\langle 1, \tau_{2} / \tau_{1}\right\rangle$ and $T^{\prime}=\mathbb{C} / \Lambda^{\prime}$. Show that the Riemann surfaces $T$ and $T^{\prime}$ are analytically isomorphic (i.e. conformally equivalent).
9. Define an equivalence relation $\sim$ on $\mathbb{C}^{*}$ by $z \sim w$ iff $z=2^{s} w$ for some $s \in \mathbb{Z}$. Show that the quotient space $R=\mathbb{C}^{*} / \sim$ has the natural structure of a compact Riemann surface, and that $R$ is analytically isomorphic to a complex torus.
10. (The identity principle for Riemann surfaces) Let $R, S$ be Riemann surfaces, and $f, g: R \rightarrow S$ be analytic maps between them. Set $E=\{z \in R: f(z)=g(z)\}$; show that either $E=R$ or $E$ contains only isolated points.
11. Let $D \subset \mathbb{C}$ be an open disc and $u$ a harmonic function on $D$. Define a complex valued function $g$ on $D$ by $g=u_{x}-i u_{y}$; show that $g$ is analytic. If $z_{0}$ denotes the centre of the disc, define a function $f$ on $D$ by

$$
f(z)=u\left(z_{0}\right)+\int_{z_{0}}^{z} g
$$

the integral being taken over the straight line segment. Show that $f$ is analytic with $f^{\prime}=g$, and that $u=\operatorname{Re} f$.
12. Suppose $u, v$ are harmonic functions on a Riemann surface $R$ and $E=\{z \in$ $R: u(z)=v(z)\}$. Show that either $E=R$, or $E$ has empty interior. Give an example to show that $E$ does not in general consist of isolated points.
13. Let $\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\}$ and $\left\{b_{1}, b_{2}, b_{3}, b_{4}\right\}$ both be sets of four distinct points in $\mathbb{C}_{\infty}$. Show that any analytic isomorphism

$$
f: \mathbb{C}_{\infty} \backslash\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\} \rightarrow \mathbb{C}_{\infty} \backslash\left\{b_{1}, b_{2}, b_{3}, b_{4}\right\}
$$

extends to an analytic isomorphism $\mathbb{C}_{\infty} \rightarrow \mathbb{C}_{\infty}$. Using your answer to Question 8 , find a necessary and sufficient condition for $\mathbb{C} \backslash\{0,1, a\}$ to be conformally equivalent to $\mathbb{C} \backslash\{0,1, b\}$, where $a, b$ are complex numbers distinct from 0 and 1.
14. Let $f(z)$ be the complex polynomial $z^{3}-z$; consider the subspace $R$ of $\mathbb{C}^{2}=\mathbb{C} \times \mathbb{C}$ given by the equation $w^{2}=f(z)$, where $(z, w)$ denote the coordinates on $\mathbb{C}^{2}$, and let $\pi: R \rightarrow \mathbb{C}$ be the restriction of the projection map onto the first factor. Show that $R$ has the structure of a Riemann surface, on which $\pi$ is an analytic map. If $g$ denotes the projection onto the second factor, show that $g$ is also an analytic map.
By deleting three appropriate points from $R$, show that $\pi$ yields a covering map from the resulting Riemann surface $R_{0} \subset R$ to $\mathbb{C} \backslash\{-1,0,1\}$, and that $R_{0}$ is analytically isomorphic to the Riemann surface (constructed by gluing) associated with the complete analytic function $\left(z^{3}-z\right)^{1 / 2}$ over $\mathbb{C} \backslash\{-1,0,1\}$.
15. Let $f(z)=\sum a_{n} z^{n}$ be a power series of radius of convergence 1 , and for $w$ in the open unit disc, set $\rho(w)$ to be the radius of convergence for the power series expansion about $w$ (so that $\rho(0)=1$ ). Show that a point $\zeta \in C(0,1)$ on the unit circle is regular if and only if $\rho(\zeta / 2)>\frac{1}{2}$. Suppose furthermore that all the $a_{n}$ are non-negative real numbers. If $\zeta \in C(0,1)$, show that $\left|f^{(r)}(\zeta / 2)\right| \leq f^{(r)}(1 / 2)$ for all $r$, and hence that $\rho(\zeta / 2) \geq \rho(1 / 2)$. Deduce that 1 is a singular point.

